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Roman Subdivision Domination in Graphs

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Abstract

The subdivision graph $S(G)$ of a graph G is the graph whose vertex set is the union of the set of vertices and the set of edges of G in which each edge uv is subdivided at once as uw and wv.

A Roman dominating function on a subdivision graph $S(G)=H$ is a function $f:V(H)\rightarrow\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of a Roman dominating function is the value $f(V(H))=\sum_{v\in V(H)} f(v)$. The minimum weight of a Roman dominating function on a subdivision graph H is called the Roman subdivision domination number of G and is denoted by $\gamma_{RS}(G)$.

In this paper, we study the Roman domination in subdivision graph $S(G)$ and obtain some results on $\gamma_{RS}(G)$ in terms of vertices, blocks and other different parameters of the graph G, but not the members of $S(G)$. Further we develop its relationship with other different domination parameters of G.

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Introduction

In this paper, we follow the notations of [2]. All the graphs considered here are simple, finite, nontrivial and undirected. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G respectively.

In general, we use $\langle S\rangle$ to denote the subgraph induced by the set of vertices of S. $N(v)$ and $N[v]$ denote the open and closed neighborhood of a vertex v.

The degree of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by $\deg v$. The maximum(minimum) degree among the vertices of G is denoted by $\Delta(G)(\delta(G))$. A vertex of degree one is called an end vertex and its neighbor is called a nonend vertex. A vertex v is called a cut vertex if removing it from G increases the number of components of G.

A subdivision graph $S(G)$ of a graph G is the graph whose vertex set is the union of the set of vertices and the set of edges of G in which each edge uv is subdivided at once as uw and wv.

A Roman dominating function (RDF) on a graph $G=(V,E)$ is a function $f:V\rightarrow\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$.

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The weight of a Roman dominating function is the value $f(V)=\sum_{v\in V} f(v)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number and is denoted by $\gamma_R(G)$.This concept has the historical motivation which is suggested by Ian Stewart [3] in his article 'Scientific American' entitled "Defend the Roman Empire" and is studied by Cockayne et.al[1].

A Roman dominating function $f=(V_0,V_1,V_2)$ on a graph G is a connected Roman dominating function (CRDF) on G if $\langle V_1\cup V_2\rangle$ or $\langle V_2\rangle$ is connected. The minimum weight of a CRDF is called a connected Roman domination number of G and is denoted by $\gamma_{RC}(G)$, see[7].

Analogously, we now define Roman subdivision domination number of a graph as follows.

A Roman dominating function on a subdivision graph H is a function $f:V(H)\to\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of a Roman dominating function is the value $f(V(H))=\sum_{v\in V(H)} f(v)$. The minimum weight of a Roman dominating function on a subdivision graph H is called the Roman subdivision domination number of G and is denoted by $\gamma_{RS}(G)$.

Results

We use the following results for our further results.

Theorem A[4]: For any graph G, $p-q\le\gamma(G)$.

Theorem B[8]: For any graph G, $\gamma(G)\le\frac{p}{2}$.

Theorem C[1]: For any graph G, $\gamma(G)\le\gamma_R(G)\le 2\gamma(G)$.

Theorem D[5]: For any tree T, $\gamma_R(L(T))\le\gamma_R(T)$.

Theorem E[6]: Let T be any tree with every nonend vertex of T adjacent to at least one end vertex. Then $\gamma_{Rn}(T)\le p-c+1$.

Now we list out the exact values of $\gamma_{RS}(G)$ for some standard graphs.

Theorem1:

1. For any path with at least three vertices

   $\gamma_{RS}(P_{3n})=4n$. Where $n=1,2,......$

   $\gamma_{RS}(P_{3n+1})=4n+1$.

   $\gamma_{RS}(P_{3n+2})=4n+2$.

2. For any cycle with at least three vertices

   $\gamma_{RS}(C_{3n})=4n$. Where $n=1,2,......$

   $\gamma_{RS}(C_{3n+1})=4n+2$.

   $\gamma_{RS}(C_{3n+2})=4n+3$.

3. For any wheel with at least four vertices

   $\gamma_{RS}(W_{3n+1})=4n+2$. Where $n=1,2,......$

   $\gamma_{RS}(W_{3n+2})=4n+4$.

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$\gamma_{RS}\left(W_{3n+3}\right)=4n+5$.

4. For any star $K_{1,p}$ with $p\geq 2$

   $\gamma_{RS}\left(K_{1,p}\right)=p+2$.

5. For any complete graph with at least three vertices

   $\gamma_{RS}\left(K_{p}\right)=2\left(p-1\right)$.

In the following theorem we establish the lower bound for $\gamma_{RS}\left(G\right)$.

Theorem2: Let G be any $\left(p,q\right)$ graph with $p\geq 2$ vertices. Then $p\leq\gamma_{RS}\left(G\right)$.

Proof: Let G be any $\left(p,q\right)$ graph with $p\geq 2$ vertices and $f=\left(V_0,V_1,V_2\right)$ be a γ_R-function in $S\left(G\right)$. We prove the result by induction on the number of vertices p of G.

Assume G is a graph with $p=2$ vertices. Then $\gamma_{RS}\left(G\right)=2=p$.

Assume that the result is true for all graphs with $p=k$ vertices. Then $\gamma_{RS}\left(G\right)\geq p$.

Let $D_{RS}=\left\{v_1,v_2,.........,v_n\right\}$ be the minimal Roman dominating set of $S\left(G\right)$ such that $\gamma_{RS}\left(G\right)=\left|D_{RS}\right|$. If G has $k+1=p^{'}$ vertices $(p^{'}=p+1)$, $q^{'}>q$ edges and this $\left(k+1\right)^{th}$ vertex will be adjacent to at least one vertex of G gives $G^{'}$. With this new vertex in $G^{'}$, at least one edge will be increased in $G^{'}$.

Since each edge is subdivided in $G^{'}$. So that two vertices will be increased in $S\left(G^{'}\right)$ such that one vertex $v\in V_2$ or V_1 of $S\left(G^{'}\right)$, which increases the cardinality of D_{RS}. Clearly $\gamma_{RS}\left(G\right)=\left|D_{RS}\right|+2\geq p^{'}$ or $\gamma_{RS}\left(G\right)=\left|D_{RS}\right|+1\geq p^{'}$. Hence by induction, $\gamma_{RS}\left(G\right)\geq p^{'}$.

The following lower bounds are immediate.

Corollory1: For any nontrivial connected graph G, $\alpha_0\left(G\right)+\beta_0\left(G\right)\leq\gamma_{RS}\left(G\right)$.

Corollory2: For any nontrivial connected graph G, $\alpha_1\left(G\right)+\beta_1\left(G\right)\leq\gamma_{RS}\left(G\right)$.

Theorem3: Let G be any $\left(p,q\right)$ graph. Then $p-q<\gamma_{RS}\left(G\right)$.

Proof: For any graph G, by Theorem A, $p-q\leq\gamma\left(G\right)$.

By Theorem B, $\gamma\left(G\right)\leq\frac{p}{2}$, which gives $p-q\leq\frac{p}{2}<p$.

Again by Theorem 2, $p\leq\gamma_{RS}\left(G\right)$.

Hence $p-q<\gamma_{RS}\left(G\right)$.

Theorem4: For any nontrivial connected tree with $p\geq 3$, $\gamma_{RS}\left(T\right)=2n_1+k$ where n_1 and k are the number of all nonend vertices and end vertices of T if and only if every nonend vertex of T is adjacent to at least two end vertices.

Proof: Suppose for any tree T, $\gamma_{RS}\left(T\right)=2n_1+k$. Then we consider the following cases.

Case1: Assume there exists a nonend vertex v which is adjacent to exactly one end vertex and e be an edge incident with v. Further $n_1=\left\{v_1,v_2,.........,v_n\right\}$ $n\geq 1$ be the number of all nonend vertices and

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$k=\{v_1,v_2,.........,v_i\}$ $i\geq 2$ be the number of all end vertices of T. Then $k-\{N(v)\cap k\}=V_1$ and $\{(n_1-v)\cup e\}=V_2$, which gives $\gamma_{RS}(T)<2n_1+k$, a contradiction.

Case2: Assume there exists a nonend vertex u which is not adjacent to an end vertex. Then $(n_1-u)=V_2$ and $\{u\cup k\}=V_1$, which gives $\gamma_{RS}(T)<2n_1+k$, a contradiction.

Conversely, let $f=(V_0,V_1,V_2)$ be a γ_R-function in $S(T)$. Suppose $n_1=\{v_1,v_2,.........,v_n\}$ be the number of all nonend vertices of T is adjacent to at least two end vertices and $k=\{v_1,v_2,.........,v_i\}$ be the number of all end vertices of T. Then we consider the following cases.

Case1: Suppose $H_1\subseteq n_1$ be the number of nonend vertices of T adjacent exactly two end vertices. Then $H_1=V_2$ and $k=V_1$. Or let $\{v_j\}$ be the set of nonend vertices of $S(T)$ adjacent to end vertices. Suppose there exists at least two vertices of $\{v_i\}\subset\{v_j\}$ such that $\{v_i\}\subset V_2$. Then there exists at least one vertex of $n_2\subset H_1$ such that $n_2\subset V_2$ and $\{N(n_2)\cap k\}=V_1$. Hence $\gamma_{RS}(T)=2|V_2|+|V_1|=2n_1+k$.

Case2: Suppose $H_2\subseteq n_1$ be the number of nonend vertices of T adjacent at least three end vertices. Then $H_2=V_2$ and $k=V_1$. Hence $\gamma_{RS}(T)=2|V_2|+|V_1|=2n_1+k$.

Theorem5: For any graph G, $2\gamma(G)\leq\gamma_{RS}(G)$.

Proof: By Theorem B, $\gamma(G)\leq\frac{p}{2}$, then $2\gamma(G)\leq p$.

Also from Theorem2, $p\leq\gamma_{RS}(G)$.

Hence $2\gamma(G)\leq\gamma_{RS}(G)$.

Theorem6: For any nontrivial connected tree T with $p\geq 3$, $\gamma_{RS}(T)=2\gamma_{RC}(T)$ if and only if every nonend vertex of T is adjacent to exactly two end vertices.

Proof: Let $f=(V_0,V_1,V_2)$ be a γ_R-function in $S(T)$ and $f'=(V_0',V_1',V_2')$ be a γ_{RC}-function in T. Suppose $n_1=\{v_1,v_2,.........,v_n\}$ be the set of all nonend vertices of T and $k=\{v_1,v_2,.........,v_i\}$ be the set of all end vertices of T. Then we consider the following cases.

Case1: Suppose $n_2=\{v_1,v_2,.........,v_n\}\subset n_1$ be the set of nonend vertices of T not adjacent to end vertex. Then $\forall v\in n_2$, $v\in V_1'$ in T. But $v\in V_0$ or V_1 or V_2 in $S(T)$. If $v\in V_0$, let $\{e_i;i=2\}$ be the number of edges incident with v, then for $\{e_j;j=1\}\subset\{e_i\}$, $\{e_j\}\in V_2$.If $v\in V_2$, let $\{e_k;k=2\}$ be the number of edges incident with v, then $\{e_k\}\in V_0$. If $v\in V_1$, then there exists the edges $\{e_m;m=2\}$ incident with v such that $\{e_m\}\in V_0$, which gives $2\gamma_{RC}(T)>\gamma_{RS}(T)$, a contradiction.

Case2: Suppose $n_3=\{v_1,v_2,.........,v_n\}\subseteq n_1$ be the number of nonend vertices of T adjacent to exactly one end vertex and $\{e_i\}$ be the number of all end edges of T. Then $\exists$ $\{\{n_4\},\{n_5\}\}\subset\{n_3\}$ such that $n_4=\{v_i\}$ and $n_5=\{v_j\}$, let $\{e_j\}\subset\{e_i\}$, $\forall\{e_j\}$ incident to $\forall\{v_j\}\in n_5$ such that $\{v_i\cup e_j\}=V_2$ and $\{N(v_i)\cap k\}=V_1$ in $S(T)$. But $\{n_4\cup n_5\}=V_2'$ and $V'_1=\phi$, which gives $2\gamma_{RC}(T)>\gamma_{RS}(T)$, a contradiction.

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Case3: Suppose $n_6 = \{v_1, v_2, ..., v_n\} \subseteq n_1$ be the number of nonend vertices of T adjacent to at least three end vertices. Then $\{n_6\} = V_2$ and $\{N(n_6) \cap k\} = V_1$. But $\{n_6\} = V_2^{'}$ and $V^{'}_1 = \phi$, which gives $2\gamma_{RC}(T) < \gamma_{RS}(T)$, a contradiction.

Hence all the above cases, $2\gamma_{RC}(T) < \gamma_{RS}(T)$.

Conversely, let T be a tree with every nonend vertex $\{n_1\}$ of T adjacent to exactly two end vertices. Then $\{n_1\} = V_2$ and $k = V_1$ in $S(T)$. But $\{n_1\} = V^{'}_2$ and $V^{'}_1 = \phi$, which gives $2\gamma_{RC}(T) = \gamma_{RS}(T)$. Hence the proof.

Theorem7: Let G be any graph. Then $\gamma_R(G) \le \gamma_{RS}(G)$.

Proof: By Theorem C, $\gamma_R(G) \le 2\gamma(G)$.

Also by Theorem 5, $2\gamma(G) \le \gamma_{RS}(G)$.

Hence $\gamma_R(G) \le \gamma_{RS}(G)$.

Theorem8: Let T be a tree with every nonend vertex of T adjacent to at least one end vertex. Then $\gamma_{RS}(T) > p - c + 1$ where c be the number of cut vertices of T.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_R-function in $S(T)$. Suppose $n_1 = \{v_1, v_2, ..., v_n\}$ be the number of all nonend vertices adjacent to exactly one end vertex and $n_2 = \{v_1, v_2, ..., v_i\}$ be the number of nonend vertices adjacent to at least two end vertices, let c be the number of cut vertices of T and k be the number of all end vertices of T. Then we consider the following cases.

Case1: Suppose $n_1 = \phi$. Then $|n_2| = |C| = |V_2|$ and $\{N(n_2) \cap k\} = V_1$. Hence $\gamma_{RS}(T) = 2|V_2| + |V_1| > p - c + 1$.

Case2: Suppose $n_1 \ne \phi$. Then $\forall \{\{n_3\}, \{n_4\}\} \subset \{n_1\}$ such that $n_3 = \{v_i\}$ and $n_4 = \{v_j\}$, $\exists$ the set of edges $\{e_i\}$ incident with $\{v_i\} \in n_3$ such that $\{e_i \cup v_j\} = V_2$ and $\{N(v_j) \cap k\} = V_1$, which gives $\gamma_{RS}(T) = 2|V_2| + |V_1| > p - c + 1$.

Now from Theorem D, we can make $L(T) = N$ such that $\gamma_{Rl}(T)$ is a Roman line domination number.

In the following theorem, we present our concept with $\gamma_{Rl}(T)$.

Theorem9: For any tree T, $\gamma_{Rl}(T) \le \gamma_{RS}(T)$.

Proof: By Theorem D, $\gamma_{Rl}(T) \le \gamma_R(T)$.

Also by Theorem 7, $\gamma_R(G) \le \gamma_{RS}(G)$.

Hence $\gamma_{Rl}(T) \le \gamma_{RS}(T)$.

Again the following theorems establish the lower bound for $\gamma_{RS}(G)$

Theorem10: Let T be a tree with every nonend vertex of T adjacent to at least one end vertex. Then $\gamma_{Rn}(T) < \gamma_{RS}(T)$.

Proof: By Theorem E, $\gamma_{Rn}(T) \le p - c + 1$.

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Also by Theorem 8, $p-c+1<\gamma_{RS}(T)$.

Hence $\gamma_{Rn}(T)<\gamma_{RS}(T)$.

Theorem11: For any graph G with $p\geq 2$ vertices, $\gamma_{RC}(G)\leq\gamma_{RS}(G)$.

Proof: Let G be a graph with $p\geq 2$ vertices, $f=(V_0,V_1,V_2)$ be a γ_R-function in $S(G)$ and $f^{'}=(V_0^{'},V_1^{'},V^{'}_2)$ be a γ_{RC}-function in G. We prove the result by induction on the number of vertices p of G.

Assume G is a graph with $p=2$. Then $\gamma_{RC}(G)=2$ and $\gamma_{RS}(G)=2=\gamma_{RC}(G)$.

Assume the result is true for all graphs G with $p=k$. Then $\gamma_{RC}(G)\leq\gamma_{RS}(G)$.

Let $D_{RS}=\{v_1,v_2,........,v_n\}$ be the minimal γ_R-set of $S(G)$ and $D_{RC}=\{v_1,v_2,........,v_i\}$ be the minimal γ_{RC}-set of G such that $\gamma_{RS}(G)=|D_{RS}|$ and $\gamma_{RC}(G)=|D_{RC}|$ respectively. Suppose G has $(k+1)$ vertices and this $(k+1)^{th}$ vertex is adjacent to at least one vertex of G. Then we consider the following cases.

Case1: Suppose $(k+1)^{th}$ vertex is adjacent to at least one vertex of D_{RC} . Then we consider the following subcases.

Subcase1.1: Assume $(k+1)^{th}$ vertex is adjacent to $v\in V_2^{'}$ of D_{RC} which generates $D^{'}_{RC}$. Then $D^{'}_{RS}$ be the Roman connected dominating set of $S(G)$ such that $|D^{'}_{RC}|=|D_{RC}|$ and $|D^{'}_{RS}|>|D_{RS}|$. Hence $\gamma_{RC}(G)<\gamma_{RS}(G)$.

Subcase1.2: Assume $u\in V_1^{'}$ and v be a $(k+1)^{th}$ vertex of G. If v is adjacent to u. Then $u\in V_2^{'}$ and $v\in V_0^{'}$. But $u\in V_2$ and $v\in V_1$. Clearly $|D^{'}_{RC}|=|D_{RC}+1|$ and $|D^{'}_{RS}|>|D_{RS}|$. Hence $\gamma_{RC}(G)<\gamma_{RS}(G)$.

Subcase1.3: Assume $(k+1)^{th}$ vertex is adjacent to $(V_1^{'}\cup V_2^{'})$. Then $|D^{'}_{RC}|=|D_{RC}|$. But $|D^{'}_{RS}|>|D_{RS}|$. Hence $\gamma_{RC}(G)<\gamma_{RS}(G)$.

Case2: Suppose $(k+1)^{th}$ vertex is adjacent to at least one vertex of $V-D_{RC}$, which means $(k+1)^{th}$ vertex is adjacent to $w\in V_0^{'}$. Then $|D^{'}_{RC}|\geq|D_{RC}|$. But $|D^{'}_{RS}|>|D_{RS}|$. Hence $\gamma_{RC}(G)<\gamma_{RS}(G)$.

From all the cases, by induction we have, $\gamma_{RC}(G)\leq\gamma_{RS}(G)$.

Theorem12: For any nontrivial tree T with n blocks, $\gamma_{RS}(T)\leq 2n$.

Proof: Let T be any nontrivial tree with n blocks and $f=(V_0,V_1,V_2)$ be a γ_R-function in $S(T)$. We prove the result by induction on the number of blocks n of T.

Assume T be a tree with $n=1$ block. Then $T=P_2$. Hence $\gamma_{RS}(T)=2=2\times 1=2n$.

Assume the result is true for all the trees with $n=k$ blocks. Then $\gamma_{RS}(T)\leq 2n$.

Let $D_{RS}=\{v_1,v_2,.........,v_n\}$ be the minimal γ_R-set of $S(T)$ such that $\gamma_{RS}(T)=|D_{RS}|$. If T has $k+1$ blocks, $p'=p+1$ vertices, $q^{'}>q+1$ edges and this $(k+1)^{th}$ block is adjacent to at least one block of T. With this new block, one vertex and one edge will be increased in T. Since each edge of T subdivides at once, hence two vertices will be increased in corresponding $S(T)$. So that cardinality of D_{RS} will be increased in $S(T)$. Hence $\gamma_{RS}(T) \leq 2n$.

Finally we obtain the Nordhauss-Gaddum type results.
Theorem13: For any graph G with $p \geq 2$,

1. $\gamma_{RS}(G) + \gamma_{RS}(\overline{G}) \leq 3p - 1$.
2. $\gamma_{RS}(G) \cdot \gamma_{RS}(\overline{G}) \leq (p+1)^2$

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